Internal Report ITESRE 286/2000

GAUSSIAN BEAM FOR CORRUGATED FEEDHORNS

F. VILLA

Istituto TESRE, CNR, Bologna, Italy

July 2000

-

GAUSSIAN BEAM FOR CORRUGATED FEEDHORNS

F. VILLA

Istituto TESRE, CNR, Bologna, Italy

SUMMARY — Gaussian beam theory has been succesfully used to characterize quasi-optical components at mm- and Infrared-wavelenght. The gaussian modelization is useful for predicting diffraction and near-field propagation effects in a simple way. Expecially for corrugated feeds the gaussian propagation can be applied to predict the radiation pattern and the phase center location as well as the coupling between feeds and reflectors. In this report a summary of the gaussian beam theory, with particular attention to some useful formulas for corrugated horn, is reported.

1 Introduction

Corrugated feedhorns can be considered as Gaussian Beam lunchers with high efficiency provided that the HE_{11} hybrid mode propages inside the horn. The gaussian beam propagation theory can be used to characterise this kind of horns in a very simple way. With the gaussian representation of feeds some commercial optical software, not specially developed for mm-wave application, can be used for predicting the performances of mm-wave telescopes where the diffraction effects are not negligible, without using time-consuming calculation methods like Physical Optics. The wave or *Helmholtz* equation

$$\left(\nabla^2 + k^2\right)\psi = 0 \tag{1}$$

is the starting point to derive the gaussian beam theory. Introducing the so called *Paraxial Approximation* on the quasi-plane wave solution of the equation (1) a *Paraxial Equation* is found and solution are the *Gaussian Modes*. In Section 2 the fundamental gaussian beam representation has been found following standard methods (see Goldsmith, 1998).

The application of the theory to the corrugated feed horns has been extensively reported in Section 3; some parctial formulas have been derived, specially devoted to the calculation of basic horn parameters like the edge taper, the angular resolution, the phase center location and the beam pattern shape at the far-field.

2 Gaussian Beam Propagation — Mathematics: The Fundamental Gaussian Beam

The wave equation for an electric field component,$E(x, y, z)$, of an electromagnetic wave is

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} + \left(\frac{2\pi}{\lambda}\right)^2 E = 0 \tag{2}$$

provided that the wave is propagating in an unifor medium and assuming a time dependence $e^{-i\omega t}$. Solution of this equation are the plane wave if the field amplitudes are independent on the position. Considering a more general solution with the amplitude dependent on spatial coordinates,

$$E(x, y, z) = E_0(x, y, z)e^{-ikz} \tag{3}$$

the wave equation is thus traslated to the following aplitude field equation:

$$\frac{\partial^2 E_0}{\partial x^2} + \frac{\partial^2 E_0}{\partial y^2} + \frac{\partial^2 E_0}{\partial z^2} - 2ik\frac{\partial E_0}{\partial z} = 0 \tag{4}$$

Additional hypothesis on the variation of the field aplitude E_0 can be introduced (Paraxial Approximation) [1]

$$\frac{\Delta\left(\frac{\partial E_0}{\partial z}\right)}{\Delta z} \cdot \lambda << \frac{\partial E_0}{\partial z} \tag{5}$$

The wave equation is now

$$\frac{\partial^2 E_0}{\partial x^2} + \frac{\partial^2 E_0}{\partial y^2} - 2ik\frac{\partial E_0}{\partial z} \tag{6}$$

and is called the *Paraxial wave equation.* Solutions are called *Gaussian Beam modes.* In cylindrical coordinates, (r, ϕ, z), the equation (6) is

$$\frac{\partial^2 E_0}{\partial r^2} + \frac{1}{r}\frac{\partial E_0}{\partial r} + \frac{1}{r}\frac{\partial^2 E_0}{\partial \phi^2} - 2ik\frac{\partial E_0}{\partial z} = 0 \tag{7}$$

and, under the hypothesis of cylindrical symmetry, a solution of this equation will be

$$E_0(r, z) = A(z)e^{-\frac{ikr^2}{2q(z)}} \tag{8}$$

where $A(z)$ and $q(z)$ are two complex function of z only. $A(z)$ and $q(z)$ are related one to each other by

$$\frac{dA}{A} = -\frac{dq}{q} \tag{9}$$

and $q(z)$, as well as $A(z)$ is related to two fundamental parameters of the gaussian beam: the radius of curvature of the beam, R, and the beam radius, w, which is the value of the radius at which the field is $1/e$ of the on-axis field. It is now possible to write the solution in a more convenient form

$$E(r, z) = \left(\frac{w_0}{w}\right) e^{-\frac{r^2}{w^2} - ikz - \frac{i\pi r^2}{\lambda R} + i\phi_0} \tag{10}$$

[1] the paraxial approximation consists to assume that the variation of the field amplitude is small along the propagation direction in a distance comparable to the wavelenght, and small to the variation on the orthogonal plane.

where

$$R = z + \frac{1}{z}\left(\frac{kw_0^2}{2}\right)^2 \tag{11}$$

$$w = w_0\left[1 + \left(\frac{2z}{kw_0^2}\right)^2\right]^{\frac{1}{2}} \tag{12}$$

$$\tan\phi_0 = \frac{2z}{kw_0^2} \tag{13}$$

The equation (10) is the simplest solution or *Fundamental Gaussian Beam*. The parameter w_0 is called the *beam waist* and is the beam radius at $z = 0$, where the curvature of the beam is $R = \infty$ and the wave front is a plane. The beam waist is a parameter that fully characterize the beam.

It is obviously possible to find solution of the paraxial equation with a more complex variations of the field perpendicular to the z-axis (propagation direction). These complex solution are called *High order Gaussian Beams*.

3 Gaussian Beam Propagation — Physics: The Corrugated Feedhorns

To represent the feedhorns as a gaussian lunchers only the *Beam Waist* (value and position on space) has to be determined. This parameter can be determined by the geometry of the corrugated horn. However, some properties on the far-field beam shape can be used to calculate the value of w_0 as well.

3.1 From the theory of Corrugated horns

The first method is based on the inverse formulas of equations (11) and (12):

$$w_0 = \frac{w}{\left[1 + \left(\frac{kw^2}{2R}\right)^2\right]^{\frac{1}{2}}} \tag{14}$$

$$z = \frac{R}{1 + \left(\frac{2R}{kw^2}\right)^2} \tag{15}$$

applied to the theory of corrugated feedhorn under the propagation of the HE_{11} hybrid mode only (balanced hybrid condition).

The aperture field of a balanced horn is characterised by the J_0 bessel function with the first null at $r = a$. The gaussian fit of this function is well represented by the fundamental mode with

$$w_a = 0.644 \cdot a \tag{16}$$

at the horn aperture plane. The curvature radius of the beam at the aperture is set by the horn length L from the cone apex to the aperture plane; thus we can substitute these quantities on equations (14) and (15), obtaining

$$w_0 = \frac{0.644 \cdot a}{\left[1 + \left(\frac{k(0.644 \cdot a)^2}{2L}\right)^2\right]^{\frac{1}{2}}} = \frac{0.644 \cdot a}{\left[1 + \left(\frac{\pi(0.644 \cdot a)^2}{\lambda L}\right)^2\right]^{\frac{1}{2}}} \tag{17}$$

$$z = \frac{L}{1 + \left(\frac{2L}{k(0.644 \cdot a)^2}\right)^2} = \frac{L}{1 + \left(\frac{\lambda L}{\pi(0.644 \cdot a)^2}\right)^2} \tag{18}$$

and then,

$$w_0 = \frac{0.644 \cdot a}{[1 + 1.69763 \cdot s^2]^{\frac{1}{2}}} \tag{19}$$

$$z = \frac{L}{1 + \left[\frac{0.58906}{s^2}\right]} \tag{20}$$

where we defined

$$s = \frac{a^2}{\lambda L} \tag{21}$$

In fact s is a typical parameter for horns because is the phase error at the aperture. It can be calculated by considering the wave front tangent to the aperture plane and originating at the cone apex. The wave path at the horn side is longer than the path at the horn axis by a quantity (for small angle horns)

$$\Delta = R - L = R - R \cdot \cos(\theta_f) = a \cdot \frac{1 - \cos(\theta_f)}{\sin(\theta_f)} = a \cdot \tan\left(\frac{\theta_f}{2}\right) \simeq a \cdot \frac{a}{2L} \tag{22}$$

$$\delta\phi = k \cdot \Delta = \frac{2\pi}{\lambda} \cdot \frac{a^2}{2L} = \frac{\pi \cdot a^2}{\lambda L} = \pi \cdot s \tag{23}$$

where θ_f is the flare angle of the horn. z in equation (15) is the distance between the phase center and the horn aperture. For a open-ended corrugated cylindrical waveguide $s = 0$ and thus the phase center is at the aperture $z = 0$ and the waist is the beam radius at the aperture, $w_0 = w_a = 0.644 \cdot a$.

3.2 Edge Taper and FWHM at the Far-Field

From equation (10) we can calculate the ratio of the amplitude at different r beam radius:

$$\frac{|E(r_1, z)|}{|E(r_2, z)|} = e^{\left\{\frac{r_2^2 - r_1^2}{w^2}\right\}} \tag{24}$$

or in terms of power:

$$\frac{P(r_1, z)}{P(r_2, z)} = \frac{|E(r_1, z)|^2}{|E(r_2, z)|^2} = e^{2\left\{\frac{r_2^2 - r_1^2}{w^2}\right\}} \tag{25}$$

A convenient choice is to refer the power at the on-axis level, setting $r_2 = 0$ and $r_1 = r_{ET}$. We obtaint then,

$$ET = \frac{P(r, z)}{P(0, z)} = e^{-2\left(\frac{r_{ET}}{w}\right)^2} \tag{26}$$

This ratio, in the antenna jargon is called Edge Taper, ET, a parameter used to match the feed with a reflector. Keeping the reflector diameter constant, the ET represent the illumination of the reflector, which directly controls the angular resolution (FWHM) and the spillover radiation.

The Edge Taper is often written in dB:

$$ET(\text{dB}) = 10 \cdot \log(ET) \tag{27}$$

Keeping the ET constant means that the ratio r_{ET}/w is constant as function of z. At the Far-Field, $z \to \infty$, the angle at which the ET can be seen is

$$\theta_{ET} = \lim_{z\to\infty}\left[\arctan\left(\frac{r_{ET}}{z}\right)\right] = \lim_{z\to\infty}\left[\arctan\left(\frac{r_{ET}}{w}\cdot\frac{w}{z}\right)\right] = \arctan\left(B\cdot\frac{\lambda}{\pi w_0}\right) \tag{28}$$

where

$$B = \frac{r_{ET}}{w} = \sqrt{-\frac{1}{2}\ln ET} = \sqrt{-0.11513 \cdot ET(\text{dB})} \tag{29}$$

The Far-Field angle at which the field falls to $1/e$ (-8.686dB) can be easly calculated substituting $r_{ET} = w$ or $B = 1$:

$$\theta_0 = \arctan\left(\frac{\lambda}{\pi w_0}\right) \tag{30}$$

Also the FWHM (Full Width Half Maximum) can be easily calculated by setting the $ET = -3dB$, and then $B = 0.5877$, obtaining

$$\text{FWHM} = 2\cdot\arctan\left(0.5877\cdot\frac{\lambda}{\pi w_0}\right) \tag{31}$$

The Normalised Far-Field pattern as function of θ (angle from the boresight direction) is calculated from equation (26) with $z = \infty$ and $t = \tan\theta = r/z$, remembering that at the Far-Field $z/w = (\pi w_0)/\lambda$:

$$P(t) = e^{-\frac{t^2}{2\left(\frac{\lambda}{2\pi w_0}\right)^2}} \tag{32}$$

or in terms of θ

$$P(\theta) = e^{-\frac{(\tan\theta)^2}{2\left(\frac{\lambda}{2\pi w_0}\right)^2}} \tag{33}$$

Let $\sigma = \frac{\lambda}{2\pi w_0}$, the pattern in dB will be:

$$P^{\text{dB}}(\theta) = -\frac{10}{2\ln 10}\cdot\frac{(\tan\theta)^2}{\sigma^2} = -2.1715\cdot\frac{(\tan\theta)^2}{\sigma^2} \tag{34}$$

4 Conclusion

In this report a first order treatement of corrugated feedhorns with gaussian beam modelisation has been investigated. Basic formulas, based on the fundamental gaussian mode, have been derived. These formulas can be used to characterise corrugated feedhorns starting from

- geometrical parameters of the corrugated horns, like horn lenght and aperture diameter
- far field beam pattern shape

From the geometrical parameter the beam waist and the location of the phase center below the horn aperture as well can be obtained. From the beam pattern shape, unfortunately, only the beam waist can be recovered. However since the first approach is applicable only for pure hybrid mode corrugated horn, it is not suitable to calculate the gaussian beam parameters for few-mode horns, like profiled or dual-profiled horns. On the contrary, the second approach can be used, in principle, for any kind of horns.

More accurate analysis, not illustrated here, can be performed considering the multi-mode gaussian expansion of the far field pattern to represent with very high accuracy the feedhorns.

5 References

- P.F. Goldsmith, 1998, *Quasioptical System: Gaussian Beam Quasioptical Propagation and Applications*, IEEE Press/Chapman & Hall Publishers Series on Microwave Technology and RF

- A.D. Olver, P.J.B. Clarricoats, A.A. Kishk and L. Shafai, 1994 *Microwave Horns and Feeds*, IEE Electromagnetic wave series 39

- J. Tuovinen, 1992, *Accuracy of a Gaussian Beam*, IEEE Trans. AP, Vol. 40, No. 4, pp. 391 – 398

- R.J. Wylde, D.H. Martin, *Gaussian Beam-Mode Analysis and Phase-Centers of Corrugated Feed Horns*, IEEE Trans. on Microwave and Techniques, Vol. 41, No. 10, pp. 1691 – 1699